Method for driven-dissipative problems: Keldysh-Heisenberg equations

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Driven-dissipative systems have recently attracted great attention due to the existence of novel physical phenomena with no analog in the equilibrium case. The Keldysh path-integral theory is a powerful tool to investigate these systems. However, it has still been a challenge to study strong nonlinear effects implemented by recent experiments, since in this case the photon number is few and quantum fluctuations play a crucial role in the dynamics of the system. Here we develop an approach for deriving exact steady states of driven-dissipative systems by introducing the Keldysh partition function in the Fock-state basis and then mapping the standard saddle-point equations into Keldysh-Heisenberg equations. We take the strong Kerr nonlinear resonators with and without the nonlinear driving as two examples to illustrate our method. It is found that, in the absence of the nonlinear driving, the exact steady state obtained does not exhibit bistability and agrees well with the complex $P$-representation solution. While in the presence of the nonlinear driving, the multiphoton resonance effects are revealed and are consistent with the qualitative analysis. Our method provides an intuitive way to explore a variety of driven-dissipative systems especially with strong correlations.

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I. INTRODUCTION

In recent years, the driven-dissipative systems have gotten a lot of attention both theoretically and experimentally. In these systems, the nonlinear interactions can be significantly enhanced by controlling both the driving and dissipation processes. For example, strong optical nonlinearities at the single-photon level have already been observed in cavity quantum electrodynamics (QED) [1,2], Rydberg atomic systems [3–5], optomechanical systems [6], and superconducting circuit QED systems [7–12]. These advances in experimental methods have greatly promoted the development of quantum metrology, quantum information, and quantum optical devices [13,14]. On the other hand, they also provide good platforms for studying novel nonequilibrium physical phenomena, such as the dynamical critical phenomena [15–17], time crystals [18], and driven-dissipative strong correlations [19,20]. In this context, how to understand the nonlinear effects in nonequilibrium phenomena has become an important topic.

The Keldysh functional integral formalism in the coherent-state basis is a general approach to study nonequilibrium physics [21]. This technique provides a well-developed toolbox of perturbation techniques to study the nonlinear effects [22–24]. For example, in some systems such as the polariton condensates [15,16] and atomic ensembles in cavities [25–30], the single-particle actions are quadratic and the diagrammatic perturbation theory, based on Wick’s theorem, can be performed. However, for coherently driven systems such as optomechanical systems [31–34], the single-particle actions are no longer quadratic and Wick’s theorem cannot be applied directly. Fortunately, when the coherent driving is strong and the nonlinear interaction is weak, the mean photon number is large and the standard saddle-point approximation can be well introduced. In this approach, the mean values of operators are mainly determined by the classical path, which satisfies the saddle-point equations, and quantum fluctuations are treated as perturbations [22–24]. However, recent researches have focused on the strongly nonlinear effects at the level of individual photons, which are a benefit for processing quantum information [14]. Experimentally, these require the systems to be weakly driven and the nonlinear interactions to be strong. As a result, the mean photon number is few and quantum fluctuations play a crucial role in the dynamics of the system. This indicates that the standard saddle-point approximation is not reasonable.

To solve this crucial problem, we develop the Keldysh path-integral theory in the Fock-state basis, from which the standard saddle-point equations are mapped into quantum Hamiltonian equations named as Keldysh-Heisenberg equations. As a result, the exact steady states induced by the quantum fluctuation effect can be well derived. We take the strong Kerr nonlinear resonators with and without nonlinear driving as two examples to illustrate our method. It is found that, in the absence of the nonlinear driving, the exact steady state obtained does not exhibit bistability and agrees well with the complex $P$-representation solutions. While in the presence
of the nonlinear driving, the multiphoton resonance effects are revealed and are consistent with the qualitative analysis. Our method offers an effective way to explore a variety of driven-dissipative systems, especially with strongly correlated photons, based on the powerful toolbox of quantum field theory.

II. STANDARD SADDLE-POINT APPROXIMATION

We begin to consider a dissipative Kerr nonlinear resonator with a coherent driving term, in which the Hamiltonian is written as \( \hat{H} = \Delta_\omega \hat{a} \hat{a}^\dagger + \chi \hat{a} \hat{a}^2 \hat{a}^\dagger + i \Omega (\hat{a}^\dagger - \hat{a}) \),

\[
H = \Delta_\omega \hat{a} \hat{a}^\dagger + \chi \hat{a} \hat{a}^2 \hat{a}^\dagger + i \Omega (\hat{a}^\dagger - \hat{a}),
\]

where \( \hat{a}^\dagger \) is the annihilation (creation) operator of the resonator, \( \Delta_\omega = \omega_c - \omega_p \) is the detuning with \( \omega_c \) and \( \omega_p \) being respectively the frequencies of the resonator and driving field, \( \chi \) is the Kerr nonlinearity, and \( \Omega \) is the driving amplitude. We assume that the resonator is coupled to a zero-temperature bath. Therefore, the dynamics of such a system are described by the Lindblad master equation \[35,36]

\[
\frac{d}{dt} \rho(t) = \mathcal{L}[\rho(t)] = -i[H, \rho(t)] + \gamma \{\hat{a} \rho(t) \hat{a}^\dagger + \hat{a}^\dagger \rho(t) \hat{a} - \hat{a} \hat{a}^\dagger \rho(t) / 2\}
\]

where \( \rho(t) \) is the density matrix, \( \mathcal{L} \) is the Liouville super-operator, \( \gamma \) is the one-photon decay rate, and \( \{\hat{a} \rho(t) \hat{a}^\dagger + \hat{a}^\dagger \rho(t) \hat{a} - \hat{a} \hat{a}^\dagger \rho(t) / 2\} \) is the standard dissipator in the Lindblad form. This Lindblad master equation can be investigated by the Keldysh nonequilibrium quantum field theory \[22–24\], in which the evolution takes place along the closed time contour.

We suppose \( |\alpha\rangle \) as a coherent state, which is the eigenstate of the annihilation operator \( \hat{a} \) with the complex eigenvalue \( \alpha \) (i.e., \( \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \)). Note that the Keldysh close contour can be divided into a sequence of infinitesimal time steps, as shown in Fig. 1(a). Then the completeness relation in terms of the coherent state, \( \tilde{1}_{\text{coh}} = \int \langle \alpha^* \alpha | e^{i S}| \alpha \rangle | \alpha \rangle \), is inserted in between consecutive time steps \[24\]. In this coherent-state basis, the partition function, which corresponds to the Lindblad master equation (2), is given by

\[
Z = \int \mathcal{D} [a_+, a_-] \exp (i S),
\]

where \( + \) and \( - \) denote the forward and backward branches and the action

\[
S = \int_{-\infty}^{\infty} dt \left\{ a_+^\dagger (i \partial_t - \Delta_\omega) a_+ - \chi a_+ a^2 a^\dagger - i \Omega (a^\dagger_+ - a_-) \\
- a_+^\dagger (i \partial_t - \Delta_\omega) a_- + \chi a_+ a^2 a^\dagger + i \Omega (a_-^\dagger - a_-) \\
- i \gamma a_+ a^\dagger_+ + i \frac{\Omega}{2} (a_- a_+^\dagger + a_+ a_-^\dagger) \right\}.
\]

Note that the operators acting on the left- and right-hand sides of the density matrix in Eq. (2) are corresponding to the fields on the forward (+) and backward (−) time branches in the Keldysh formalism \[24\]. This leads to characteristics of the Keldysh functional integral with the doubling of degrees of freedom. Therefore, the time evolution can be interpreted as occurring along the closed Keldysh contour.

It is more convenient to discuss Eq. (4) in the Keldysh basis,

\[
a_{cl} = \frac{1}{\sqrt{2}} (a_+ + a_-), \quad a_q = \frac{1}{\sqrt{2}} (a_+ - a_-),
\]

where \( a_{cl} \) and \( a_q \) are the classical and quantum fields \[22–24\]. After a straightforward calculation, the action is rewritten as

\[
S = \int_{-\infty}^{\infty} dt \left\{ a_{cl}^\dagger (i \partial_t - \Delta_\omega) a_{cl} + a_q^\dagger (i \partial_t - \Delta_\omega) a_q \\
- i \frac{\gamma}{2} (a_{cl} a_q - a_q a_{cl}^\dagger) + i \gamma a_q a_q^\dagger - i \sqrt{2} \Omega (a_q^\dagger - a_q) \\
- \chi (a_{cl}^2 a_{cl} a_q + a_q a_{cl}^2 a_q^\dagger + a_{cl}^2 a_q a_q^\dagger + a_q^2 a_{cl} a_q^\dagger) \right\}.
\]

Note that in the presence of the coherent driving (\( \Omega \neq 0 \)) the first two lines of Eq. (6) are not quadratic. Therefore, we cannot directly apply the diagrammatic perturbation theory, which is based on Wick’s theorem, to calculate the nonlinear term. Fortunately, when the coherent driving is strong and the nonlinear interaction is weak, the mean photon number circulating inside the resonator is large and the light field behaves as a semiclassical field \[35\]. In such a case, the saddle-point approximation can be well used to investigate the dynamics of the system \[22–24\]. As show in Fig. 1(b), the mean values of operators are mainly determined by the classical path and quantum fluctuations are treated as perturbation. The classical path is determined by the principle of least action:

\[
\frac{\delta S}{\delta a_{cl}^\dagger} = 0, \quad \frac{\delta S}{\delta a_q} = 0,
\]

which lead to two saddle-point equations

\[
i \partial_t a_q = \frac{1}{2} (2 \Delta_\omega + i \gamma) a_q + \chi (2a_{cl}^2 a_{cl} a_q + a_q^2 a_{cl}^2 + a_{cl}^2 a_q^2),
\]

\[
i \partial_t a_{cl} = i \sqrt{2} \Omega - i \gamma a_q + \frac{1}{2} (2 \Delta_\omega - i \gamma) a_{cl}.
\]
The correspondence between these two methods is shown in a diagram (Fig. 2), where \( \Delta_c / \gamma = -0.25 \). The red solid lines are the stable solutions of Eq. (11), while the red dash-dotted line is its unstable solution. These mean-field solutions reflect the optical bistability phenomenon. The blue dashed line is the exact steady-state solution from Eq. (26), which has considered the quantum fluctuation effect.

\[
\langle \hat{a}^+ \hat{a} \rangle = \langle a_0 \rangle^2 = \frac{4\Omega^2}{4(\Delta_c + 2\chi |a_0|^2)^2 + \gamma^2}. \tag{11}
\]

This solution is identical to the mean-field solution of the steady state [35,36]. In fact, the saddle-point approximation is equivalent to the mean-field approach, named the linearization approximation, in quantum optics [35]. In the spirit of the linearization approximation, the operator \( \hat{a} \) can be split into an average amplitude and a fluctuation term, i.e., \( \hat{a} \rightarrow \langle \hat{a} \rangle + \delta \hat{a} \), where \( \langle \hat{a} \rangle \) is determined by the mean-field equation. The correspondence between these two methods is shown in Fig. 1(b). As pointed out in Ref. [36], when the sign of \( \Delta_c \) is opposite to that of \( \chi \), Eq. (11) may have two stable solutions (see the red lines of Fig. 2). In other words, the action of the system has two classical paths [see Fig. 1(c)]. As a result, the perturbation calculation around the classical path may be not reasonable. This phenomenon, called the optical bistability, signals the failure of both the linearization approximation and the saddle-point approximation. On the contrary, Drummond and Walls derived a complex \( P \)-representation solution for the steady state [36]. In that method, they considered the quantum fluctuation effect and found that the exact steady-state solution does not exhibit bistability.

### III. KELDYSH-HEISENBERG EQUATIONS

We note that the standard saddle-point equations are based on the coherent-state basis. The coherent state is the closest quantum-mechanical state to a classical description of the field. It is a suitable representation for optical fields when the total photon number is large and quantum fluctuations are weak [35]. Obviously, this condition is not satisfied in the bistable region. As shown in Fig. 2, in that region the coherent driving is weak and the Kerr nonlinearity is the same order as the other parameters. Therefore, the mean photon number is not so large and quantum fluctuations induced by the Kerr nonlinearity cannot be ignored. To overcome this shortcoming, we introduce the Fock state, which is the eigenstate of the photon number operator. In the Fock-state basis, we develop a method using Keldysh-Heisenberg equations that governs the quantum fluctuation effect.

In the Fock-state basis, the completeness relation inserted in between consecutive time steps of the Keldysh close time contour becomes \( \hat{1}_F = \sum_n |n\rangle \langle n| \) (see Fig. 3). In this case, the Keldysh partition function for stationary states reads (see Appendix A for details)

\[
Z = \text{Tr}\{\exp(i\hat{S})\}, \tag{12}
\]

where \( \text{Tr} \) denotes the trace operation which connects the two time branches, giving rise to the closed Keldysh contour [24]. \( \hat{S} \) is the quantum action (like a time-evolution operator), where

\[
\hat{S} = - \int_{-\infty}^{+\infty} \hat{H} \text{d}t. \tag{13}
\]

In Eq. (13), \( \hat{H} \) is a generalized Hamiltonian operator. As shown in Appendix A, \( \hat{H} \) consists of operators acting on different branches of the Keldysh close time contour. For the driven-dissipative Kerr resonator described in Eq. (2), the generalized Hamiltonian operator has the form

\[
\hat{H} = \Delta_c \hat{a}^+_c \hat{a}_+ + \chi \hat{a}^+_c \hat{a}_- \hat{a}^+_c \hat{a}_- + i\Omega(\hat{a}^+_c - \hat{a}_+) - \Delta_c \hat{a}^+_a \hat{a}^- + \chi \hat{a}^+_a \hat{a}^- \hat{a}^+_a \hat{a}^- - i\Omega(\hat{a}^+_a - \hat{a}_a) + i\gamma \hat{a}_a \hat{a}^- - i\gamma \hat{a}^+_a \hat{a}_a - \hat{a}^+_a \hat{a}^- - \frac{\gamma}{2} \hat{a}^+_a \hat{a}^- - \hat{a}^- \hat{a}^+_a - \hat{a}^- \hat{a}^+_a, \tag{14}
\]

where \( \hat{a}_a \) (\( \hat{a}^+_a \)) are the annihilation (creation) operators and the subscript \( + \) (\( - \)) means that the operator only acts on the forward (backward) time branch. These operators obey the commutation relations: \( [\hat{a}_a, \hat{a}^+_b] = [\hat{a}_a, \hat{a}^-_b] = 1 \) and \( [\hat{a}_a, \hat{a}^-_a] = 0 \). Note that \( \hat{H} \) is a non-Hermitian operator and,
we see that the degrees of freedom are doubled in the Keldysh formalism. Therefore, the Hilbert space of $\hat{H}$ is a doubling Hilbert space $\mathcal{H}_+ \otimes \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are the Hilbert spaces corresponding to $\hat{a}_+$ and $\hat{a}_-$, respectively. Comparing Eq. (13) with Eq. (4), it can be found that we only need to replace the complex variable $a_+ (a_-)$ by the corresponding operator $\hat{a}_+$ ($\hat{a}_-$) and omit the derivative with respect to time.

The time evolution of $\hat{a}_\pm$ can be governed by (see Appendix B for details)

$$i \frac{d}{dt} \hat{a}_\pm = [\pm \hat{a}_\pm, \hat{H}].$$

We further define $\hat{a}_q = (\hat{a}_+ + \hat{a}_-)/\sqrt{2}$ and $\hat{a}_q = (\hat{a}_+ - \hat{a}_-)/\sqrt{2}$ as the annihilation operators of the classical and quantum fields, respectively. Immediately, these operators obey the commutation relations: $[\hat{a}_q, \hat{a}_q^\dagger] = [\hat{a}_q, \hat{a}_q^\dagger] = 1$ and $[\hat{a}_q, \hat{a}_q^\dagger] = 0$. And the quantum action in Eq. (13) is changed to $S = -\int_{-\infty}^{+\infty} \hat{H} dt = -\int_{-\infty}^{+\infty} (\hat{H}_t + \hat{H}_l) dt$, where

$$\hat{H}_t = i\sqrt{2\Omega} \hat{a}_q^\dagger + \frac{1}{2}(2\Delta_e - i\gamma) \hat{a}_q \hat{a}_q^\dagger + \chi (\hat{a}_q^\dagger \hat{a}_q + \hat{a}_q \hat{a}_q^\dagger - 1) \hat{a}_q \hat{a}_q^\dagger,$n

$$\hat{H}_l = -i\sqrt{2\Omega} \hat{a}_q - i\gamma \hat{a}_q \hat{a}_q^\dagger + \frac{1}{2}(2\Delta_e + i\gamma) \hat{a}_q^\dagger \hat{a}_q + \chi (\hat{a}_q^\dagger \hat{a}_q + \hat{a}_q \hat{a}_q^\dagger - 1) \hat{a}_q^\dagger \hat{a}_q.$$

Through these considerations, the Hilbert space of $\hat{H}$ is changed to $\mathcal{H}_q \otimes \mathcal{H}_{cl}$, where $\mathcal{H}_q$ and $\mathcal{H}_{cl}$ are the Hilbert spaces corresponding to $\hat{a}_q$ and $\hat{a}_{cl}$, respectively. And the time evolutions of $\hat{a}_q$ and $\hat{a}_{cl}$ can be governed by the following equations:

$$i \frac{d}{dt} \hat{a}_q = [\hat{a}_q, \hat{H}_l] + \chi (\hat{a}_q^\dagger \hat{a}_q + \hat{a}_q \hat{a}_q^\dagger - 1) \hat{a}_q \hat{a}_q^\dagger,$n

$$i \frac{d}{dt} \hat{a}_{cl} = [\hat{a}_{cl}, \hat{H}_t] + \chi (\hat{a}_{cl}^\dagger \hat{a}_{cl} + \hat{a}_{cl} \hat{a}_{cl}^\dagger - 1) \hat{a}_{cl} \hat{a}_{cl}^\dagger.$$

Since Eqs. (18) and (19) are formally similar to the Heisenberg equations for an equilibrium system, we can call them Keldysh-Heisenberg equations. Interestingly, it is easy to verify that the Keldysh-Heisenberg equations can also be obtained by quantizing the semiclassical saddle-point equations. In contrast to the standard saddle-point equations, Keldysh-Heisenberg equations can completely capture the information induced by quantum fluctuations. Therefore, we use them to obtain the exact steady-state solution.

We assume the steady-state wave function as $|\Psi_0\rangle$, which is a vector in the doubling Hilbert space $\mathcal{H}_q \otimes \mathcal{H}_{cl}$. Note that, in the steady state, the expectation values of operators do not evolve over time; i.e., $i \frac{d}{dt} \langle \Psi_0 | \hat{a}_q | \Psi_0 \rangle = 0$ and $i \frac{d}{dt} \langle \Psi_0 | \hat{a}_{cl} | \Psi_0 \rangle = 0$. Using these conditions and Eqs. (18) and (19), we obtain the following equations:

$$0 = \langle \Psi_0 | \hat{a}_q, \hat{H}_l | \Psi_0 \rangle = \langle \Psi_0 | \frac{1}{2}(2\Delta_e + i\gamma) \hat{a}_q \hat{a}_q^\dagger + \chi (2\hat{a}_q^\dagger \hat{a}_q + \hat{a}_q \hat{a}_q^\dagger) | \Psi_0 \rangle,$n

$$0 = \langle \Psi_0 | \hat{a}_{cl}, \hat{H}_t | \Psi_0 \rangle = \langle \Psi_0 | i\sqrt{2\Omega} - i\gamma \hat{a}_q + \frac{1}{2}(2\Delta_e - i\gamma) \hat{a}_q^\dagger \hat{a}_q + \chi (2\hat{a}_{cl}^\dagger \hat{a}_{cl} + \hat{a}_{cl} \hat{a}_{cl}^\dagger) | \Psi_0 \rangle.$$
which is equivalent to the complex $P$-representation solution in Ref. [36]. In Fig. 2, we plot the steady-state mean photon number as a function of the coherent driving amplitude $\Omega$. Obviously, the exact steady state does not exhibit bistability (see the blue dashed line).

**IV. NONLINEAR DRIVING CASE**

In this section, we extend our method to the two-photon nonlinear driving case implemented recently in superconducting quantum circuits [40]. The effective Hamiltonian reads

$$\hat{H} = \Delta_i \hat{a}_i^\dagger \hat{a}_i + \chi \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i + i \Omega (\hat{a}_i^\dagger - \hat{a}_i) + \frac{1}{2} (\lambda \hat{a}_i^\dagger \hat{a}_i^\dagger + \lambda^* \hat{a}_i^\dagger \hat{a}_i),$$

where $\Lambda$ is the complex amplitude of the two-photon driving term. The Lindblad master equation becomes

$$\frac{d}{dt} \rho(t) = -i [\hat{H}, \rho(t)] + \gamma D[\hat{a}] \rho(t) + \kappa D[\hat{a}^\dagger] \rho(t),$$

where $\kappa$ is the two-photon loss rate.

In the presence of the two-photon driving and loss terms, we rewrite $\dot{H} = \dot{H}_1 + \dot{H}_1$ as

$$\dot{H}_1 = \frac{1}{2} (2 \Delta_i - i \gamma) \hat{a}_i^\dagger \hat{a}_i + \chi (\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i) - 2 \Delta_i \hat{a}_i^\dagger \hat{a}_i + i \sqrt{2} \Omega \hat{a}_i^\dagger - i \frac{\kappa}{2} (\hat{a}_i^\dagger \hat{a}_i^\dagger - \hat{a}_i^\dagger \hat{a}_i) + \lambda \hat{a}_i^\dagger \hat{a}_i^\dagger,$$

$$\dot{H}_1 = \frac{1}{2} (2 \Delta_i + i \gamma) \hat{a}_i \hat{a}_i^\dagger + \chi (\hat{a}_i \hat{a}_i^\dagger + \hat{a}_i \hat{a}_i^\dagger) - 2 \Delta_i \hat{a}_i \hat{a}_i^\dagger - i \sqrt{2} \Omega \hat{a}_i + i \frac{\kappa}{2} (\hat{a}_i \hat{a}_i^\dagger - \hat{a}_i \hat{a}_i^\dagger) + (i \gamma + 2 \kappa) \hat{a}_i \hat{a}_i^\dagger + \lambda \hat{a}_i \hat{a}_i^\dagger.$$  

It is also easy to verify that $\dot{H}[\Psi_0] = 0$ and $\langle \Psi_0 | \dot{H} | \Psi_0 \rangle = 0$. Finally, using $\dot{H}[\Psi_0] = 0$ we get a recursion relation for the expansion coefficient as

$$\langle \dot{a}_i | \hat{a}_i \rangle = \frac{1}{N} \sum_{m=0}^{+\infty} \frac{1}{m!} \mathcal{F}_m \mathcal{F}_m^{+ \dagger},$$

where $\mathcal{F}_m = (\lambda)^{m+k} 2 \mathcal{F}_1[-(m + k), \gamma; z; 2]$. It can be verified that Eq. (35) is equivalent to the complex $P$-representation solution in Ref. [41]. Using Eq. (35), we can study the influence of different driving processes on the nonlinear effects. For example, we consider the multiphoton resonances in the weak driving regime, which are easy to observe in experiments and can be used to measure the photon-photon interactions [42]. In this situation, the mean photon number is small and the mean-field approach is not reasonable. We first make a qualitative prediction from the Hamiltonian (27). When the energy of $n$ incident photons is equivalent to the energy of $n$ photons inside the resonator, that is, $n \omega_p = n \omega_c + n \chi(n-1)$, the absorption of $n$ pumping photons is favored. Expressed in terms of the detuning $\Delta_c = \omega_c - \omega_p$, this relation reads $\Delta_c / \chi = -(n-1)$. On the other hand, the parity of $n$ depends on the driving processes. In the absence of the one-photon driving ($\Omega = 0$ and $\Delta \neq 0$), an even number of pumping photons is favored ($n$ is even) and $\Delta_c / \chi = -1, -3, -5, \ldots$, while in the presence of both the one- and two-photon driving ($\Omega \neq 0$ and $\Delta \neq 0$), $n$ can be any integer greater than 0 and $\Delta_c / \chi = 0, -1, -2, -3, -4, \ldots$ In Fig. 4, we plot the steady-state mean photon number as a function of the detuning $\Delta_c / \chi$, based on Eq. (35). This figure shows clearly that, in the absence of the one-photon pumping (see the blue dashed line), the photon resonances arise around $\Delta_c / \chi = -1$ and $-3$, while in the presence of both the one- and two-photon driving (see the red solid line), the photon resonances arise around $\Delta_c / \chi = 0, -1, -2, -3$. These results are consistent with the qualitative analysis.
between strongly correlated systems and the environment. Whereas for the coherent quantum-absorber method, it is difficult to construct auxiliary resonators to simulate the coupling between strongly correlated systems and the environment. Instead, the Keldysh-Heisenberg equations could solve these complex problems since it can be easy to combine with powerful tools of quantum field theory, such as the linked-cluster expansion approach and the renormalization group method [22]. For example, we just need to divide the quantum action $S$ as $S = S_0 + S_i = -\int_{-\infty}^{+\infty} H_0 dt - \int_{-\infty}^{+\infty} H_i dt$, where $H_0$ is the solvable part of the generalized Hamiltonian $\hat{H}$ and $H_i$ is the perturbation part. Then we can take the steady-state wave function of $H_0$ as the unperturbed steady state and calculate the influence of $H_i$ by the perturbation theory. We also noticed that in the previous literature of strong-correlated systems, the Keldysh path functional integral is in the coherent-state basis [17,46,47]. Therefore, the quantum fluctuation effects are still not been fully studied. In the near future, we hope our method can be extended to explore nonequilibrium phenomena induced by quantum fluctuations.

V. CONCLUSIONS

In summary, we have established the Keldysh path-integral theory in the Fock-state basis, from which the Keldysh-Heisenberg equations are successfully introduced. In contrast to the standard saddle-point equations, these quantum operator equations can well describe the quantum fluctuation effect and thus present the exact steady-state solutions. We have also considered two examples of the driven-dissipative Kerr nonlinear resonators with and without the two-photon nonlinear driving. Our results agree well with the qualitative analysis and those obtained by the complex-$P$-representation method [36,41] and the coherent quantum-absorber method [43,44].

Before ending this paper, we compare our method with the complex-$P$-representation method [36] and the coherent quantum-absorber method [43,44], both of which have also considered the quantum fluctuation effect. For the complex-$P$-representation method, an operator master equation can be formally solved by the perturbation theory. We also noticed that for the coherent quantum-absorber method, both of which have also considered two examples of the driven-dissipative Kerr nonlinear resonators with and without the two-photon nonlinear driving. Our results agree well with the qualitative analysis and those obtained by the complex-$P$-representation method [36,41] and the coherent quantum-absorber method [43,44].

APPENDIX A: KELDYSH PARTITION FUNCTION IN THE FOCK-STATE BASIS

In this Appendix, we derive the Keldysh partition function in the Fock-state basis in detail. A general Lindblad master equation reads

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] + \gamma D[\hat{\sigma}]\hat{\rho}(t), \quad (A1)$$

where $\hat{H}$ is any Hamiltonian of the system. For a one- or two-photon loss process, $\hat{\sigma}$ can be chosen as $\hat{a}$ or $\hat{a}^\dagger$. Without loss of generality, we set $\hat{\sigma} = \hat{a}$ hereafter. Using the master equation (A1), the time evolution of the density matrix from $t_0$ to $t_f$ is formally solved by

$$\hat{\rho}(t_f) = e^{(t_f-t_0)\mathcal{L}}\hat{\rho}(t_0) = \lim_{N\to\infty}(1 + \delta t\mathcal{L})^N\hat{\rho}(t_0), \quad (A2)$$

where we have decomposed the time evolution into a sequence of infinitesimal steps of duration $\delta t = (t_f - t_0)/N$. We focus on a single time step, and denote the density matrix after the jth step $(t_j = t_0 + j\delta t)$ by $\hat{\rho}_j$. Then we have

$$\hat{\rho}_{j+1} = e^{\delta t\mathcal{L}}\hat{\rho}_j = (1 + \delta t\mathcal{L})\hat{\rho}_j + O(\delta t^2). \quad (A3)$$

Since the Liouville superoperator $\mathcal{L}$ acts on the density matrix “from both sides,” it is more convenient to represent the density matrix in the Keldysh closed time contour. As shown in Fig. 3 of the main text, this can be achieved by projecting the density matrix into the two time branches [24]:

$$\hat{\rho}_j \equiv \hat{P}_{+j}\hat{\rho}\hat{P}_{-j}, \quad (A4)$$

where $\hat{P}_{+j}$ ($\hat{P}_{-j}$) is the projection operator on the forward (backward) branch at the time $t_j$. Obviously, if we choose $\hat{P}$ as a unit operator of the coherent state, i.e., $\hat{P} = \hat{1}_{\text{coh}} = \int\int (d\alpha d^2\pi)e^{-i\alpha^2}|\alpha\rangle\langle\alpha|$, we can get the partition function in Sec. II [24]. Instead, here we choose $\hat{P}$ as the identity in the

FIG. 4. The steady-state mean photon number $\langle \hat{a}^\dagger\hat{a} \rangle$ as a function of the detuning $\Delta_c/\chi$, when $\Omega/\chi = 0$ (blue dashed line) and $\Omega/\chi = 0.1$ (red solid line). The other parameters are chosen as $\gamma/\chi = 0.2$, $\kappa/\chi = 0.1$, and $\Delta/\chi = 0.2$.

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Fock space, i.e., \( \hat{\rho}_c = \hat{1}_F = \sum_n |n\rangle \langle n| \). In this case, \( \hat{\rho}_j \) can be written as
\[
\hat{\rho}_j = \sum_{m,n} |m\rangle \langle m| \rho_{m,n} \langle n| = \sum_{m,n} \langle m| \hat{\rho}_j \langle n| |m\rangle \langle n| \langle n| - \langle n| l \rangle \langle l| - \langle m| \hat{\rho}_j \langle n| |m\rangle \langle n| (A5)
\]

We now consider \( \hat{\rho}_{j+1} = \sum_n \langle k| \hat{\rho}_{j+1} |l\rangle \langle l| \) in terms of the corresponding matrix element at the previous time step \( j \). Inserting Eq. (A5) into Eq. (A3), we obtain
\[
\langle k| \hat{\rho}_{j+1} |l\rangle = \sum_{m,n} [\langle k| \hat{\rho}_j |n\rangle - \langle m| \hat{\rho}_j \langle n| |m\rangle \langle n| (l) - \langle m| \hat{\rho}_j \langle n| |m\rangle \langle n| (l) - (A6)
\]

where
\[
\hat{\mathcal{H}}_{k,l,m,n} = i\langle k| \hat{\mathcal{L}} |m\rangle \langle n| |l\rangle - \langle k| \hat{\mathcal{L}} |m\rangle \langle n| |l\rangle + (A7)
\]

and the trace of \( \hat{\rho}_{j+1} \) can thus be expressed as a simple form:
\[
\text{Tr} \hat{\rho}_{j+1} = \text{Tr} \sum_{k,l,m,n} \langle k| \hat{\mathcal{L}} \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle k| \hat{\mathcal{L}} \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle m| \hat{\mathcal{L}} |n\rangle \langle l| + O(\delta t^2).
\]

By iteration of Eq. (A10), the density matrix evolves from \( \hat{\rho}(t_0) \) at \( t_0 \) to \( \hat{\rho}(t_f) \) at \( t_f = t_N \). This implies that in the limit \( N \rightarrow \infty \) (and hence \( \delta t \rightarrow 0 \)),
\[
Z_{t_0,t_f} = \exp(\delta \hat{S}) = \exp(\delta \hat{S} | t_0, t_f \rangle \langle t_0, t_f |).
\]

Finally, we perform the limit, \( t_0 \rightarrow -\infty \) and \( t_f \rightarrow +\infty \), to get the Keldysh partition function for stationary states. Since in a Markov process, the initial state in the infinite past does not affect the stationary state [24], we can ignore the boundary term, i.e., \( \hat{\rho}(t_0) \) in Eq. (A11), and obtain the final expression of the Keldysh partition function as
\[
Z = \text{Tr}[\exp(\delta \hat{S})],
\]

with the quantum action
\[
\hat{S} = -\int_{-\infty}^{+\infty} \hat{\mathcal{H}} dt.
\]

**APPENDIX B: DERIVING THE KELDYSH-HEISENBERG EQUATIONS**

In this Appendix, we derive the Keldysh-Heisenberg equations by considering the average values of operators. For example, we define \( \langle \hat{a}_+ \rangle \) as the average value of \( \hat{a}_+ \) at time \( t \). Using Eq. (A10), we find
\[
\langle \hat{a}_+ \rangle_{j+1} - \langle \hat{a}_+ \rangle_j = \text{Tr} \hat{\delta}_a \hat{\rho}_{j+1} - \text{Tr} \hat{\delta}_a \hat{\rho}_j
\]

\[
= \text{Tr} \sum_{k,l,m,n} (-(\beta t i \hat{\mathcal{H}}_{k,l,m,n} |n\rangle \langle l| + \langle k| \hat{\mathcal{L}} \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle k| \hat{\mathcal{L}} \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle m| \hat{\mathcal{L}} |n\rangle \langle l| + O(\delta t^2).
\]

Obviously, Eq. (B1) can be expressed as a simple form
\[
i \frac{d}{dt} \langle \hat{a}_+ \rangle = \{\hat{a}_+, \hat{\mathcal{H}}\}.
\]

Similarly, we also find
\[
\langle \hat{a}_- \rangle_{j+1} - \langle \hat{a}_- \rangle_j = \text{Tr} \hat{\delta}_a \hat{\rho}_{j+1} - \text{Tr} \hat{\delta}_a \hat{\rho}_j
\]

\[
= \text{Tr} \sum_{k,l,m,n} (-(\beta t i \hat{\mathcal{H}}_{k,l,m,n} |n\rangle \langle l| + \langle k| \hat{\mathcal{L}} \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle k| \hat{\mathcal{L}} \langle m| \hat{\mathcal{L}} |n\rangle \langle l| - \langle m| \hat{\mathcal{L}} |n\rangle \langle l| + O(\delta t^2).
\]

Thus, the dynamic equation of \( \langle \hat{a}_- \rangle \) can be expressed as
\[
i \frac{d}{dt} \langle \hat{a}_- \rangle = \{\hat{a}_-, \hat{\mathcal{H}}\}.
\]
Therefore, the dynamics of \( \hat{a}_+ \) and \( \hat{a}_- \) in the Heisenberg picture are described by

\[
i \frac{d}{dt} \hat{a}_k = [\pm \hat{a}_k, \hat{H}] . \tag{B5}\]

Finally, by defining \( \hat{a}_c = (\hat{a}_+ + \hat{a}_-) / \sqrt{2} \) and \( \hat{a}_q = (\hat{a}_+ - \hat{a}_-) / \sqrt{2} \), we obtain the Keldysh-Heisenberg equations in the main text, i.e.,

\[
i \frac{d}{dt} \hat{a}_q = [\hat{a}_c, \hat{H}], \quad i \frac{d}{dt} \hat{a}_c = [\hat{a}_q, \hat{H}] . \tag{B6}\]

**APPENDIX C: STEADY-STATE WAVE FUNCTION FOR THE NONLINEAR DRIVING CASE**

We present the detailed derivation of Eq. (34) of the main text. The steady-state wave function is

\[
|\Psi_0\rangle = e^{-\lambda \hat{a}_c^\dagger} |\Phi_0\rangle = \frac{1}{\sqrt{N}} e^{-\lambda \hat{a}_c^\dagger} |0\rangle_q \sum_{k=0}^{+\infty} \phi_k |k\rangle_{cl} = \frac{1}{\sqrt{N}} |0\rangle_q \sum_{j=0}^{+\infty} \frac{(-\lambda \hat{a}_c^\dagger)^j}{j!} \sum_{k=0}^{+\infty} \phi_k |k\rangle_{cl} .
\]

\[
= \frac{1}{\sqrt{N}} |0\rangle_q \sum_{j=0}^{+\infty} \frac{(-\lambda \hat{a}_c^\dagger)^j}{j!} \sum_{k=0}^{+\infty} \phi_k |k\rangle_{cl} \quad (\text{C1})
\]

Letting \( j + k = m \), we obtain

\[
|\Psi_0\rangle = \frac{1}{\sqrt{N}} |0\rangle_q \sum_{m=0}^{+\infty} \sum_{k=0}^{m} \frac{(2\lambda)^k (-\lambda)^{m-k} m! (y)_k}{\sqrt{m!(m-j)!} (z)_k} |m\rangle_{cl} \quad (\text{C2})
\]


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