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#### Abstract

In this paper, we demonstrate that the non-Hermiticity can induce reentrant localization in a generalized quasiperiodic lattice. Specifically, by considering a nonreciprocal dimerized lattice with staggered quasiperiodic disorder, we find that the localization transition can appear twice by increasing the disorder strength. We also unravel a multi-complex-real eigenenergy transition, whose transition points coincide with those in the localization phase transitions. Moreover, the impacts of boundary conditions on the localization properties have been clarified. Finally, we study the wavepacket dynamics in different parameter regimes, which offers an experimentally feasible route to detect the reentrant localization.


## 1. Introduction

As one of the most fundamental effects affecting the transport properties of quantum particles, localization has been extensively explored in condensed matter physics. Typical phenomenon is the absence of a particle's diffusion induced by random disorder, which is known as Anderson localization [1]. All states in one- and two-dimensional system with uncorrelated disorder are localized [2]. Three-dimensional systems with disorder have both localized and delocalized eigenstates, separated by a critical energy known as the single-particle mobility edge in the spectrum [3]. Quasiperiodic systems, at the interface of long-range ordered and fully disordered systems, exhibit much different localization behaviors such as a 1D localization transition at a finite (quasi-) disorder strength, critical spectra, and multifractal eigenstates. The most paradigmatic example is the Aubry-André-Harper (AAH) model [4, 5], which has been widely investigated in optical and atomic systems [6-9]. By introducing a long-range hopping term or breaking the self-duality of the AAH Hamiltonian, one can obtain single-particle mobility edge [10-12]. As a result, a single-particle intermediate phase where the localized and extended eigenstates coexist appears [13, 14].

Meanwhile, the physics of non-Hermitian systems has garnered substantial interest both theoretically and experimentally in recent years [15-17]. In general, the non-Hermiticity is achieved by introducing the complex on-site potentials or asymmetric hopping terms. They host extensive features that do not exist in Hermitian systems, such as exceptional points [18, 19], non-Hermitian skin effect [20-26], and anomalous transport behavior [27-30]. These exotic phenomena bring potential applications including lasing [31-33], sensing [34-38], and topological light modulation [39-41]. Particularly, the interplay between disorder and non-Hermiticity has been recently explored [42-59]. In non-Hermitian quasiperiodic lattice with asymmetric hopping or $\mathcal{P} \mathcal{T}$-symmetry, it is revealed that the non-Hermitian Anderson transition coincides with topological transition and complex-real energy transition [43-46]. These results have also been extended to the mobility edges [47-50] and many-body localization [51-53].

It is usually expected that, when the disorder strength is larger than a critical value, all eigenstates become localized, and they will remain so if the disorder strength is further increased. Recently, an


Figure 1. Schematic illustration of the non-Hermitian generalized quasiperiodic lattice with two sublattice ( $A$ and $B$ ) in each unit cell. $J_{1} e^{ \pm \alpha}$ and $J_{2} e^{ \pm \alpha}$ are the asymmetric intra- and inter-cell hopping strengths, and $V_{j}^{\nu}$ corresponds to the on-site quasiperiodic modulation.
interesting phenomenon termed relocalization or re-delocalization transition has been revealed in generalized AAH model [60-62]. In this paper, we demonstrate that the non-Hermiticity can induce a reentrant localization. Specifically, in a nonreciprocal dimerized lattice with staggered quasiperiodic disorder, the localization transition can appear twice if the disorder strength is persistently increased. We find such a reentrant localization transition is accompanied by a multi-complex-real transition in the energy spectrum. We also reveal that the localized states are robust to different boundary conditions whereas the extended states become skin modes under the open boundary condition (OBC). Finally, we clarify the dynamics of wavepacket in different parameter regimes. Our finding facilitates deeper understanding of non-Hermiticity, disorder, and their interplay.

## 2. Model and Hamiltonian

We consider a one-dimensional generalized AAH quasiperiodic lattice with asymmetric hopping terms. As illustrated in figure 1, the system contains two sublattice sites $A$ and $B$ in each unit cell with asymmetric intra- and inter-cell hopping strengths $J_{1} e^{ \pm \alpha}$ and $J_{2} e^{ \pm \alpha}$, respectively, which is describe by the following tight-binding Hamiltonian

$$
\begin{equation*}
\hat{H}=\sum_{j}\left[J_{1}\left(e^{\alpha} \hat{c}_{j, B}^{\dagger} \hat{c}_{j, A}+e^{-\alpha} \hat{c}_{j, A}^{\dagger} \hat{c}_{j, B}\right)+J_{2}\left(e^{\alpha} \hat{c}_{j+1, A}^{\dagger} \hat{c}_{j, B}+e^{-\alpha} \hat{c}_{j, B}^{\dagger} \hat{c}_{j+1, A}\right)\right]+\sum_{j}\left(V_{j}^{A} \hat{n}_{j, A}+V_{j}^{B} \hat{n}_{j, B}\right), \tag{1}
\end{equation*}
$$

with

$$
V_{j}^{\nu}= \begin{cases}V_{1} \cos [2 \pi \beta(2 j-1)+\varphi], & (\nu=A)  \tag{2}\\ V_{2} \cos [2 \pi \beta(2 j)+\varphi] . & (\nu=B)\end{cases}
$$

Here, $j$ represents the unit cell index, and $L=2 N$ is the length of the lattice with $N$ being the number of the unit cells. $\hat{c}_{j, \nu}^{\dagger}(\nu=A, B)$ and $\hat{c}_{j, \nu}$ are the creation and annihilation operators at site $(j, \nu)$, and $\hat{n}_{j, \nu}=\hat{c}_{j, \nu}^{\dagger} \hat{c}_{j, \nu}$ is the corresponding site number operator. The sublattice sites $A(B)$ are subject to a quasiperiodic modulation with amplitude $V_{1}\left(V_{2}\right)$ and phase $\varphi$. The modulation period is incommensurate with the lattice space and characterized by an irrational number $\beta$. Note that the non-Hermiticity in this model is controlled by the nonreciprocal strength $\alpha$. In the absence of the on-site potential $\left(V_{1}=V_{2}=0\right)$ and non-Hermiticity $(\alpha=0)$, the model corresponds to the Su-Schrieffer-Heeger model which exhibits two topologically distinct phases, trivial one with $J_{1}>J_{2}$ and nontrivial one with $J_{1}<J_{2}$, separated by a topological phase transition point at $J_{1}=J_{2}$. Such phase transition is protected by the chiral symmetry of the system, which is broken in the presence of on-site disorder. We define a quantity $\delta=J_{2} / J_{1}$ to specify the strength of hopping dimerization in equation (1). Since the parameter $\varphi$ acts as a global spatial shift of the periodic potential which does not affect the localization properties, we can simply set $\varphi=0$ without loss of generality. In the numerical calculations throughout the paper, we set $L=610, \beta=(\sqrt{5}-1) / 2$ and $J_{1}=1$, and the periodic boundary condition (PBC) is assumed unless otherwise specified.

## 3. Numerical results and analysis

Note that for $\delta=1$ and $V_{1}=V_{2}=V$, the Hamiltonian (1) reduces to the nonreciprocal AAH model [43]. Calculations based on such non-Hermitian AAH model predicts three types of transitions: (i) complex-real eigenenergy transition, (ii) localization transition, and (iii) topological phase transition characterized by the quantized jump of a spectral winding number. Similar results are also presented in the $\mathcal{P} \mathcal{T}$-symmetry AAH model $[44,63,64]$. Here, we focus on the nonreciprocal dimerized lattice $(\delta \neq 1)$ with staggered quasiperiodic disorder $\left(V_{1}=-V_{2}=V\right)[62,65]$.


Figure 2. (a) and (b) show the $\operatorname{IPR}^{(i)}$ (red dots) and $\mathrm{NPR}^{(i)}$ (blue dots) versus $V$ for $\alpha=0$ and $\alpha=0.25$, respectively. (c) and (d) show the $\langle\mathrm{IPR}\rangle$ (red-dashed curves) and the $\langle\mathrm{NPR}\rangle$ (blue-solid curves) versus $V$ for $\alpha=0$ and $\alpha=0.25$, respectively. Here, $\delta=4$. There are three regions I, II, III for the extended, intermediate, and localized phases, respectively. $V_{c 1}, V_{c 2}, V_{c 3}$, and $V_{c 4}$ denote the critical point of extended-intermediate (I-II), intermediate-localized (II-III), localized-intermediate (III-II), and intermediate-localized (II-III) transitions, respectively.

### 3.1. Non-Hermitian induced reentrant localization

For most quasiperiodic lattices hosting the single-particle mobility edge, the eigenstates are extended (localized) in the parameter regime where the disorder strength prior to (beyond) the critical phase. However, by combining together three additional elements, (a) the non-Hermitian asymmetric hopping rate, (b) the lattice dimerization, and (c) staggered disorder, the model considered here may exhibit a distinct localization behavior-the so-called reentrant localization, as detailed in the following.

To quantificationally describe the localization properties, we consider the inverse participation ratio (IPR) and the normalized participation ratio (NPR) defined respectively as

$$
\begin{align*}
\operatorname{IPR}^{(i)} & =\frac{\sum_{n}\left|u_{n}(i)\right|^{4}}{\left[\sum_{n}\left|u_{n}(i)\right|^{2}\right]^{2}},  \tag{3}\\
\operatorname{NPR}^{(i)} & =\left[L \sum_{n}\left|u_{n}(i)\right|^{4}\right]^{-1}, \tag{4}
\end{align*}
$$

where the superscript $i$ denotes the $i$ th eigenstate of the system and $n$ labels the lattice coordinate. In the large $L$ limit, the IPR tends to be zero (nonzero) and NPR tends to be nonzero (zero) for the extended states (localized states).

Figures 2(a) and (b) show the $\operatorname{IPR}^{(i)}$ and the $\operatorname{NPR}^{(i)}$ as functions of $V$ for $\alpha=0$ and $\alpha=0.25$, respectively. Apart from the extended and localized phases, an intermediate region where both $\mathrm{IPR}^{(i)}$ and $\mathrm{NPR}^{(i)}$ are merged together takes place. More interestingly, while such a region appears only once for $\alpha=0$, it repeats again for $\alpha=0.25$ by further increasing the disorder strength. This reentrant localization behavior is somehow triggered by the system non-Hermiticity. To make point clearer, we average the IPR ${ }^{(i)}$ and $\mathrm{NPR}^{(i)}$ over all eigenstates to obtain $\langle\mathrm{IPR}\rangle$ and $\langle\mathrm{NPR}\rangle$, where the symbol $\langle\ldots\rangle$ indicates the averaged value.

In figures 2(c) and (d), we plot the $\langle\mathrm{IPR}\rangle$ and the $\langle\mathrm{NPR}\rangle$ as functions of $V$ for $\alpha=0$ and $\alpha=0.25$, respectively. Inspecting the values of $\langle\mathrm{IPR}\rangle$ and $\langle\mathrm{NPR}\rangle$, three distinct phases can be clearly identified-extended phase with a vanishing $\langle\mathrm{IPR}\rangle$ and a finite $\langle\mathrm{NPR}\rangle$, localized phase with a finite $\langle\mathrm{IPR}\rangle$ and a vanishing $\langle\mathrm{NPR}\rangle$, and an intermediate phase where both $\langle\mathrm{IPR}\rangle$ and $\langle\mathrm{NPR}\rangle$ are finite (shaded region). From this definition, the intermediate phase is nothing but a parameter region where the localized and extended states coexist. The difference between Hermitian case and non-Hermitian case becomes distinct by increasing $V$. For the Hermitian case (figure 2(c)), all the states remain localized after the first localization transition, whereas the intermediate phase for the non-Hermitian case (figure 2(d)) are located within two separate regions $V_{c 1}<V<V_{c 2}$ and $V_{c 3}<V<V_{c 4}$, indicating that the localization transition occurs twice.


Figure 3. Phase diagrams of $\eta$ (a) and $\langle r\rangle$ (b) in the $\alpha-V$ plane with $\delta=4$.

The reentrant localization feature can also be conveniently captured by the quantity [14]

$$
\begin{equation*}
\eta=\log _{10}[\langle\mathrm{IPR}\rangle \times\langle\mathrm{NPR}\rangle], \tag{5}
\end{equation*}
$$

which can be used to clearly distinguish the intermediate region from the extended or localized regions in the phase diagram. In figure $3(\mathrm{a})$, we plot $\eta$ in the $\alpha-V$ plane with $\delta=4$. It is to be seen that for $\alpha<\alpha_{c 1}$, the extended and localized phases are respectively located at weak and strong disorder strength $V$, with an intermediate region sitting in between. Above the first critical point $\alpha_{c 1}(\approx 0.156)$, an additional lobe of intermediate phase appears for larger $V$, signifying the onsite of the reentrant localization. Further increasing the asymmetric hopping such that $\alpha>\alpha_{c 2}(\approx 0.31)$, the two lobes of the intermediate phase merge.

Another quantity to characterize the localization property is the level statistic [66-68], whose main feature is encoded in the adjacent gap ratio,

$$
\begin{equation*}
r_{i}=\frac{\min \left(s_{i+1}, s_{i}\right)}{\max \left(s_{i+1}, s_{i}\right)} \tag{6}
\end{equation*}
$$

where $s_{i+1}=\operatorname{Re}\left(E_{i+1}\right)-\operatorname{Re}\left(E_{i}\right)$ denotes the level spacing between the real part of the $(i+1)$ th and $i$ th eigenenergies [49,56,57]. The average of $r_{i}$ which goes over all the eigenenergies gives rise to $\langle r\rangle=\sum_{n} r_{n} / L$. As shown in figure 3(b), in the extended phase, $\langle r\rangle$ approaches zero, whereas $\langle r\rangle \simeq 0.4$ in the localized phase, and the intermediate phase appears in the region $0<\langle r\rangle<0.4$. The obtained phase boundary is consistent with the results shown in figure 3(a).

We emphasize that the reentrant localization is not exclusive to the non-Hermitian dimerized lattice. In fact, Hermitian system may exhibit similar behavior under some fine-tuned parameter settings [62]. The asymmetric hopping can, however, largely extend the parameters regime where the reentrant localization occurs, and therefore offers an additional experiment knob detecting richer physics. In the following, we first study the spectra structure of our system and then demonstrate the dynamics of wavepacket to characterize the reentrant localization.

### 3.2. Spectra structure and skin effect

In general, the non-Hermitian Hamiltonians possess complex energy spectra, whose distribution on the complex plane is tightly related to the localization property of the system. Especially for systems with asymmetric hopping, the localization prohibits the imaginary parts of complex eigenenergies and the extended-localized phase transition is usually accompanied by a real-complex transition of the energy spectra.

Figure 4(a) shows the energy spectra of the system for different $V$ with $\{\alpha=0.25, \delta=4\}$ on the complex plane. It can be seen that, while all the eigenenergies are complex (purely real) in the extended phase (localized phase), only part of the eigenenergies are real in the intermediate phase. In figures $4(\mathrm{~b})$ and (c), we respectively plot the real and imaginary parts of the eigenenergies, together with their corresponding $\operatorname{IPR}^{(i)}$, as functions of $V$. It is clear that the critical points separating the complex-real, real-complex, and complex-real transitions coincide with those of the localization transition, and moreover, within each localized phases, the imaginary parts of the eigenenergies completely disappear as expected. These results can be further confirmed by investigating the following ratio,

$$
\begin{equation*}
f_{\mathrm{Im}}=D_{\operatorname{Im}} / D \tag{7}
\end{equation*}
$$

where $D_{\operatorname{Im}}$ is the number of eigenenergies with nonzero imaginary parts, and $D$ is the total number of eigenenergies. As shown in figure $4(\mathrm{~d}), f_{\mathrm{Im}}$ is depicted as a function of $\alpha$ and $V$. It is found that the curves (white-dashed curves) separating different values of $f_{\text {Im }}$ perfectly reproduces the boundaries between phases


Figure 4. (a) Complex energy spectrum $E$ of the non-Hermitian Hamiltonian (1) with various $V$. (b) and (c) show the real and the imaginary part of the eigenvalue spectra versus $V$ for the system with $\delta=4$. The color code indicates the values of $\operatorname{IPR}^{(i)}$. (d) $f_{\text {Im }}$ as a function of $\alpha$ and $V$ with the white-dashed curve being the phase boundary. (e) Phase diagram of the reentrant localization. The other parameters are the same as those in figure 2(d).
with different localization properties. To be specific, $f_{\text {Im }}=1\left(f_{\text {Im }}=0\right)$ corresponds to extended (localized) phase, and $0<f_{\text {Im }}<1$ is related to intermediate phase. The spectra distribution on the complex plane, together with the localization properties, can be summarized in a single diagram shown in figure 4(e).

Non-Hermitian systems with asymmetric hopping may exhibit distinct behaviors under different boundary conditions. As a notable example, under OBC, all the extended eigenstates turn into skin modes whose spatial distributions are mainly localized at the sample boundaries. In this section, we investigate the effect of boundary conditions on the energy spectra and localization properties of the considered model. Figure 5 shows the energy spectra in the complex plane for different $V$ under both PBC and OBC. As shown in figure 5(a), in the extended phase, the eigenenergies for PBC and OBC are completely separated, and moreover, the skin modes are created under OBC (see the inset of figure 5(a)). Results for intermediate phase are displayed in figures 5 (b) and (d). It can be seen that, some of the eigenenergies are complex, whose eigenstates are extended (skin modes) for PBC (OBC), and the others with localized eigenstates are purely real. The purely real ones for PBC and OBC overlap in the complex plane, whereas those with nonzero imaginary parts remain separated. The results for localized phase are plotted in figures 5(c) and (e). It is clear that the energy spectra under PBC and OBC are the same, implying the localized states are insensitive to the boundary conditions. From the above discussion, it is found that knowledge about localization properties can be easily gained by inspecting the spectra structure in the complex plane and vice versa. Of course, the information of the reentrant localization is also encoded in the spectra structure.

### 3.3. Wavepacket dynamics

Same as the Hermitian case, the localization property of the non-Hermitian system can also be effectively captured by the wavepackets dynamics $[69,70]$. We then expect that the wavepacket dynamics can help us detect the reentrant localization.

We study the spreading of wavepacket initialized at the center of the lattice. The state vector at time $t$ is written as $|\psi(t)\rangle=\mathrm{e}^{-\mathrm{i} \hat{H} t}|\psi(0)\rangle$, where the initial state assumes $|\psi(0)\rangle=\left|\delta_{n, 0}\right\rangle$. Since $\hat{H}$ is non-Hermitian here, the Schrödinger equation itself does not preserve the norm of $|\psi(t)\rangle$. An additional normalization operation acts as

$$
\begin{equation*}
|\psi(t)\rangle=\exp (-i \hat{H} t)|\psi(0)\rangle / \| \exp (-i \hat{H} t)|\psi(0)\rangle \|, \tag{8}
\end{equation*}
$$

is needed in the numerical simulation $[51,54,55]$. Figure 6 shows the time evolution of the wavepacket for different values of $V$ under OBC. As shown in figure 6(a), when the wavepacket is initialized in the extended phase, it is ballistically transported to one side of the lattice due to the presence of skin modes on the corresponding boundary. In the localized phase, on the other hand, the wavepacket is frozen into its initial position without spreading. The wavepacket propagating in the intermediate phase may exhibit some singular behavior distinct from those in extended and localized phases. While initially localized for a short time, it starts to spread and move to the right side of the lattice at certain time.


Figure 5. Energy spectra in the complex energy plane for system with various $V$ under PBC (blue circles) and OBC (red dots). (a) $V=0.8$, (b) $V=2.0$, (c) $V=3.0$, (d) $V=4.5$, and (e) $V=5.5$. The insets plot the distribution of eigenstates corresponding to the eigenvalues labeled by the arrows under OBC. The other parameters are the same as those in figure 2(d).


Figure 6. Evolution of wavepacket initially localized at the center site of the lattice for various $V$ with $L=800$. (a) $V=0.8$, (b) $V=2.0$, (c) $V=3.0$, (d) $V=4.5$, and (e) $V=5.5$. The other parameters are the same as those in figure 2(d).

Different types of the wavepacket spreading dynamics can be quantitatively classified by the spreading velocity

$$
\begin{equation*}
v \sim \frac{\sigma(t)}{t} \tag{9}
\end{equation*}
$$

where $\sigma(t)=\sqrt{\sum_{n} n^{2}\left|\psi_{n}(t)\right|^{2}}$ is the mean square displacement. In general, for long enough time, $\sigma(t)$ obeys the power law $\sigma(t) \sim t^{\gamma}$, where $\gamma$ is the quantum diffusion exponent. Here $\gamma=1, \gamma=1 / 2$, and $\gamma=0$ corresponds to the ballistic transport, diffusive transport, and localization, respectively. In figure 7, we plot $v$ as a function a $V$ (red-circles curve). It is clear that $v$ approaches 0 in the localized phases and takes finite values otherwise. Another related observable is the survival probability, defined as

$$
\begin{equation*}
F(l)=\lim _{t \rightarrow \infty} \sum_{-l \leqslant n \leqslant l}\left|\psi_{n}(t)\right|^{2}, \tag{10}
\end{equation*}
$$

which describes the probability of finding the excitation within the region $(-l, l)$ in the long time limit. Figure 7 (blue-squares curve) plots $F(l=10)$ as a function of $V$ under OBC. It is shown that $F(l=10)$ vanishes in the extended and intermediated phases, whereas in the localized phase $F(l=10) \simeq 1$.


Figure 7. Survival probability $F$ (blue-squares curve) and spreading velocity $v$ (red-circles curve) as a function of $V$ for $t=150$. The other parameters are the same as those in figure 2(d).


Figure 8. Upper panel: the scheme of the electric circuit to simulate Hamiltonian (1). $C$ and $C_{a(b)}$ denote the capacitance of the capacitor and INIC for a capacitor, respectively. These two electric devices connect two neighboring nodes, which simulates the nonreciprocal hopping. The on-site potential is simulated by grounding each nodes with two electric devices with capacitance $C_{j, A(B)}$ and $C_{j, A(B)}^{\prime}$, respectively. Lower panel: the structure of the INIC.

Therefore, the survival probability provide a feasible experiment route to observe the non-Hermitian induced reentrant localization.

## 4. Possible experimental realization

We propose a possible experimental scheme by employing electrical circuits to simulate the reentrant localization, as shown in figure 8 . The nonreciprocal hoppings between neighboring sites can be simulated by capacitor and the negative impedance converter with current inversion (INIC) [47, 71]. The impedance of the INIC will be changed from positive to negative if the current orientation through the device is reversed, or vice versa. The on-site potential is realized by grounding each nodes with appropriate electric devices. According to Kirchhoff's law, we have $\mathbf{I}=\mathbf{J V}$, where the vector components of $\mathbf{I}$ and $\mathbf{V}$ correspond to the current and voltage of each node, respectively. We can use the circuit Laplacian $\mathbf{J}$ to simulate the Hamiltonian. The circuit Laplacian in figure 8 reads

$$
\mathbf{J}=i \omega\left(\begin{array}{ccccc}
D_{1, A} & -\left(C-C_{a}\right) & 0 & \ldots & 0  \tag{11}\\
-\left(C+C_{a}\right) & D_{1, B} & -\left(C-C_{b}\right) & \ldots & 0 \\
0 & -\left(C+C_{b}\right) & D_{2, A} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\ldots & \ldots & \ldots & \ldots & D_{L, B}
\end{array}\right)
$$



Figure 9. (a) and (b) The $\langle\mathrm{IPR}\rangle$ (a) and $\langle\mathrm{NPR}\rangle$ (b) as a function of $V$ with $L=754,1220,1974$ and 2584. (c) and (d) The $\langle\mathrm{IPR}\rangle$ (c) and $\langle\mathrm{NPR}\rangle$ (d) as a function of $1 / L$ with $V=2.0,3.0,4.5$ and 5.5.
where $D_{1, A}=C-C_{a}-C_{1, A}^{\prime}+C_{1, A}, D_{L, B}=C+C_{b}-C_{L, B}^{\prime}+C_{L, B}$, and $D_{j, A}=2 C+C_{b}-C_{a}-C_{j, A}^{\prime}+C_{j, A}$ and $D_{j, B}=2 C+C_{a}-C_{b}-C_{j, B}^{\prime}+C_{j, B}$ for $1<j<L$. Here, $\omega$ is the frequency of the current. To establish the mapping between the parameters of two Hamiltonian systems, equations (1) and (11), we set $C \pm C_{a}=J_{1} e^{ \pm \alpha}$ and $C \pm C_{b}=J_{2} e^{ \pm \alpha}$. To model the diagonal terms, we can take $C_{1, A}^{\prime}=C-C_{a}$, $C_{L, B}^{\prime}=C+C_{b}, C_{j, A}^{\prime}=2 C+C_{b}-C_{a}(1<j<L), C_{j, B}^{\prime}=2 C+C_{a}-C_{b}(1<j<L), C_{j, A}=V_{1} \cos [2 \pi \beta$ $(2 j-1)+\varphi]$ and $C_{j, B}=V_{2} \cos [2 \pi \beta(2 j)+\varphi]$. The energy spectrum is obtained from the admittance spectrum of the circuit and the wavepacket dynamics can be detected by measuring the voltage response of each node.

## 5. Discussion

In the above discussions, we have considered the staggered disorder with $V_{1}=-V_{2}=V$. However, this is not a strict requirement for the existence of reentrant localization. Actually, the reentrant localization still exists as long as $V_{1} V_{2}<0$. Another point is the finite size effects. To further confirm that the reentrant localization is independent of the system size, we compute the $\langle\mathrm{IPR}\rangle$ and $\langle\mathrm{NPR}\rangle$ for different $L$, as shown in figures 9(a) and (b). It is clear that the second intermediate region exists even in the large $L$ limit. The finite size analysis of $\langle\mathrm{IPR}\rangle$ and $\langle\mathrm{NPR}\rangle$ are shown in figures 9 (c) and (d), respectively. These results indicate the reentrant behavior of the localization transition.

Finally, the non-Hermiticity can also be introduced by the complex on-site potential [44, 49]. This can be achieved by changing $\varphi$ to $\varphi+i h$ in equation (2). The non-Hermitian parameter $h$ significantly reduces the reentrant localization regime, which has been discussed in reference [65]. This kind of non-Hermiticity cannot induce the reentrant localization.

## 6. Conclusion

To conclude, we have studied the localization transition in a non-Hermitian dimerized lattice with staggered quasiperiodic disorder. We have demonstrated that the non-Hermiticity with asymmetric hopping can induce a reentrant localization for a wide range of parameters. By analyzing the spacial distribution of wave functions and the corresponding eigenenergies, we found that such reentrant localization transition is accompanied by a multi-complex-real transition. We have also investigated the impacts of boundary conditions on the localization properties. It is found that the extended states turn into skin modes under OBC, whereas the localization states are robust to different boundary conditions. Finally, we studied the wavepacket dynamics in different parameter regimes to characterize and detect the reentrant localization.

Our results can be examined in some experimentally feasible settings such as electric circuits [71], cold atom systems [42, 72], and photonic systems [73].

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## Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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## References

[1] Anderson P W 1958 Phys. Rev. 1091492
2] Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 Phys. Rev. Lett. 42673
[3] Evers F and Mirlin A D 2008 Rev. Mod. Phys. 801355
[4] Aubry S and André G 1980 Ann. Israel Phys. Soc. 3133
[5] Harper P G 1995 Proc. Phys. Soc. A 68874
[6] Lahini Y, Pugatch R, Pozzi F, Sorel M, Morandotti R, Davidson N and Silberberg Y 2009 Phys. Rev. Lett. 103013901
[7] Kraus Y E, Lahini Y, Ringel Z, Verbin M and Zilberberg O 2012 Phys. Rev. Lett. 109106402
[8] Roati G, D’Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 Nature 453895
[9] Xue P, Qin H, Tang B and Sanders B C 2014 New J. Phys. 16053009
[10] Biddle J and Das Sarma S 2010 Phys. Rev. Lett. 104070601
[11] Li X, Li X and Das Sarma S 2017 Phys. Rev. B 96085119
[12] Lüchen H P, Scherg S, Kohlert T, Schreiber M, Bordia P, Li X, Das Sarma S and Bloch I 2018 Phys. Rev. Lett. 120160404
[13] Kohlert T, Scherg S, Li X, Lüschen H P, Das Sarma S, Bloch I and Aidelsburger M 2019 Phys. Rev. Lett. 122170403
[14] Li X and Das Sarma S 2020 Phys. Rev. B 101064203
[15] Konotop V V, Yang J and Zezyulin D A 2016 Rev. Mod. Phys. 88035002
[16] Ödemir s̨, Rotter S, Nori F and Yang L 2019 Nat. Mater. 18783
[17] Bergholtz E J, Budich J C and Kunst F K 2021 Rev. Mod. Phys. 93015005
[18] Regensburger A, Bersch C, Miri M A, Onishchukov G, Christodoulides D N and Peschel U 2012 Nature 488167
[19] Miri M A and Alù A 2019 Science 363 eaar7709
[20] Kunst F K, Edvardsson E, Budich J C and Bergholtz E J 2018 Phys. Rev. Lett. 121026808
[21] Yao S and Wang Z 2018 Phys. Rev. Lett. 121086803
[22] Yokomizo K and Murakami S 2019 Phys. Rev. Lett. 123066404
[23] Zhang K, Yang Z and Fang C 2020 Phys. Rev. Lett. 125126402
[24] Okuma N, Kawabata K, Shiozaki K and Sato M 2020 Phys. Rev. Lett. 124086801
[25] Li L, Lee C H, Mu S and Gong J 2020 Nat. Commun. 115491
[26] Xiao L, Deng T, Wang K, Zhu G, Wang Z, Yi W and Xue P 2020 Nat. Phys. 16761
[27] Longhi S 2020 Phys. Rev. Lett. 124066602
[28] Yi Y and Yang Z 2020 Phys. Rev. Lett. 125186802
[29] Liu C H, Zhang K, Yang Z and Chen S 2020 Phys. Rev. Res. 2043167
[30] Lee C H, Li L, Thomale R and Gong J 2020 Phys. Rev. B 102085151
[31] Hodaei H, Miri M-A, Heinrich M, Christodoulides D N and Khajavikhan M 2014 Science 346975
[32] Parto M et al 2018 Phys. Rev. Lett. 120113901
[33] Malzard S and Schomerus H 2018 New J. Phys. 20063044
[34] Hodaei H, Hassan A U, Wittek S, Garcia-Gracia H, El-Ganainy R, Christodoulides D N and Khajavikhan M 2017 Nature 548187
[35] Chen W, Kaya Özdemir ş, Zhao G, Wiersig J, Yang L, Wiersig J and Yang L 2017 Nature 548192
[36] Budich J C and Bergholtz E J 2020 Phys. Rev. Lett. 125180403
[37] McDonald A and Clerk A A 2020 Nat. Commun. 115382
[38] Chen C, Jin L and Liu R-B 2019 New J. Phys. 21083002
[39] Zhao H, Qiao X, Wu T, Midya B, Longhi S and Feng L 2019 Science 3651163
[40] Weidemann S, Kremer M, Helbig T, Hofmann T, Stegmaier A, Greiter M, Thomale R and Szameit A 2020 Science 368311
[41] Jin L, Wang P and Song Z 2017 New J. Phys. 19015010
[42] Gong Z, Ashida Y, Kawabata K, Takasan K, Higashikawa S and Ueda M 2018 Phys. Rev. X 8031079
[43] Jiang H, Lang L J, Yang C, Zhu S L and Chen S 2019 Phys. Rev. B 100054301
[44] Longhi S 2019 Phys. Rev. Lett. 122237601
[45] Schiffer S, Liu X-J, Hu H and Wang J 2021 Phys. Rev. A 103 L011302
[46] Cai X 2021 Phys. Rev. B 103014201
[47] Zeng Q B and Xu Y 2020 Phys. Rev. Res. 2033052
[48] Liu T, Guo H, Pu Y and Longhi S 2020 Phys. Rev. B 102024205
[49] Liu Y, Jiang X-P, Cao J and Chen S 2020 Phys. Rev. B 101174205
[50] Liu Y, Wang Y, Liu X J, Zhou Q and Chen S 2021 Phys. Rev. B 103014203
[51] Hamazaki R, Kawabata K and Ueda M 2019 Phys. Rev. Lett. 123090603
[52] Zhai L J, Yin S and Huang G Y 2020 Phys. Rev. B 102064206
[53] Tang L Z, Zhang G Q, Zhang L F and Zhang D W 2021 Phys. Rev. A 103033325
[54] Longhi S 2021 Phys. Rev. B 103054203
[55] Xu Z and Chen S 2021 Phys. Rev. A 103043325
[56] Wang C and Wang X R 2020 Phys. Rev. B 101165114
[57] Hamazaki R, Kawabata K, Kura N and Ueda M 2020 Phys. Rev. Res. 2023286
[58] Luo X, Ohtsuki T and Shindou R 2021 Phys. Rev. Lett. 126090402
[59] Kawabata K and Ryu S 2021 Phys. Rev. Lett. 126166801
[60] Goblot V et al 2020 Nat. Phys. 16832
[61] Zhai L J, Huang G Y and Yin S 2021 Phys. Rev. B 104014202
[62] Roy S, Mishra T, Tanatar B and Basu S 2021 Phys. Rev. Lett. 126106803
[63] Zhou L and Han W 2021 Chinese Phys. B 30100308
[64] Xia X, Huang K, Wang S and Li X 2021 arXiv:2105.12640
[65] Jiang X-P, Qiao Y and Cao J-P 2021 Chinese Phys. B 30097202
[66] Atas Y Y, Bogomolny E, Giraud O and Roux G 2013 Phys. Rev. Lett. 110084101
[67] Goldsheid I Y and Khoruzhenko B A 1998 Phys. Rev. Lett. 802897
[68] Chalker J T and Mehlig B 1998 Phys. Rev. Lett. 813367
[69] Larcher M, Laptyeva T V, Bodyfelt J D, Dalfovo F, Modugno M and Flach S 2012 New J. Phys. 14103036
[70] Xu Z, Huangfu H, Zhang Y and Chen S 2020 New J. Phys. 22013036
[71] Helbig T et al 2020 Nat. Phys. 16747
[72] Li L, Lee C H and Gong J 2020 Phys. Rev. Lett. 124250402
[73] Weidemann S, Kremer M, Longhi S and Szameit A 2021 Nat. Photon. 15576

